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Discrete representations for the deformed $su(1,1)$ algebra via the magnetic monopole harmonics

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Abstract

It is well known that the magnetic quantum number m of monopole harmonics describes quantization corresponding to the z-component of the angular momentum operator in the framework of $su(2)$ symmetry algebra. Here, it is shown that the azimuthal quantum number l allocates itself a ladder symmetry by the operators which are described in terms of l. Furthermore, quantization of both quantum numbers l and m leads to extracting positive and negative irreducible discrete representations of the deformed $su(1,1)$ algebra for $l-m$ and $l+m$, respectively. It will also be shown in detail how they in turn lead to the spectrum-generating algebra for all monopole harmonics.

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(Some figures in this article are in colour only in the electronic version)

1. Introduction

The azimuthal and magnetic quantum numbers l and m via the monopole (generalized) harmonics describe the surface (angular) part of wavefunctions corresponding to the motion of a charged particle in the presence of a magnetic monopole which was first studied by Dirac [1], and then by Wu and Yang [2, 3]. These generalized harmonics are characterized by the magnetic charge q and should be considered as a complete set of orthonormal bases for the square integrable sections of complex line bundles over the sphere. Furthermore, it has been shown that three independent generalizations of the spherical harmonics on sphere, i.e. Wigner $\mathcal{D}$ functions, spin-weighted spherical harmonics and monopole harmonics are equivalent to each other [4–8]. Many studies have been performed on the magnetic monopole by other authors, see [9] and references therein. Some of these studies have recently drawn attention to the infinite dimensional representations of the rotation group by solutions of Dirac's magnetic monopole [10, 11]. The authors have shown that the nonunitary representations of the rotation group and the generalization of the Dirac quantization condition can be obtained from the magnetic monopole solutions. Also, realization of finite and unitary irreducible representations of $su(2)$ Lie algebra via the m index of monopole harmonics is a problem that arises alternately in the works on mathematical physics (see, for example, [3, 7, 8]). Both of these are strong motivations which lead us to enquire into their other internal symmetries.

Simultaneous ladder symmetry with respect to two parameters of special functions has lead to a deep and extended understanding of solvability, supersymmetry, representation theory and coherent states for the one-, two- and three-dimensional models. Realization of this idea, whose formulation has also been performed for the associated Jacobi functions [12], provides a rich algebraic structure for them and their corresponding differential equation. The ladder symmetry with respect to the azimuthal quantum number l of monopole harmonics has been considered in [13] with the limitation $0 \leqslant m \leqslant l$ on the magnetic quantum number. In this paper, in addition to extending the formulation to $-l \leqslant m \leqslant l$, the infinite ladder symmetry with respect to l together with $su(2)$ algebra symmetry for m are applied to show that the deformed $su(1,1)$ algebra can also irreducibly be represented by using monopole harmonics. Here, we will show that the appropriate Hilbert subspaces of all monopole harmonics represent the deformed $su(1,1)$ algebra by simultaneous shift operators of both quantum labels l and m for given values of $l-m$ and $l+m$, respectively. Therefore, a new spectrum-generating algebra as an internal symmetry for the Hilbert space corresponding to the monopole harmonics is constructed.

2. Monopole harmonics $Y_{lm}^{N(S);q}(\theta,\phi)$

The second-order differential equation corresponding to the Jacobi polynomials of degree n in the interval $-1 \leqslant x \leqslant +1$ and its solutions in terms of the Rodriguez formula are well known as [12]

$$\begin{aligned}
&(1-x^2)P_n''^{(\alpha,\beta)}(x) - [\alpha-\beta+(\alpha+\beta+2)x]P_n'^{(\alpha,\beta)}(x) \\
&\qquad + n(\alpha+\beta+n+1)P_n^{(\alpha,\beta)}(x) = 0, \\
&P_n^{(\alpha,\beta)}(x) = \frac{a_n(\alpha,\beta)}{(1-x)^\alpha(1+x)^\beta}\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^n((1-x)^{n+\alpha}(1+x)^{n+\beta}) \qquad \text{with} \quad \alpha,\beta > -1,
\end{aligned} \tag{1}$$

in which $a_n(\alpha,\beta)$'s are the normalization coefficients. If we define the functions $\mathcal{P}_{l,m}^{(q)}(x)$ as

$$\mathcal{P}_{l,m}^{(q)}(x) := \frac{a_{l,m}(q)}{a_{l-m}(m-q,m+q)}\sqrt{\frac{2l+1}{2}}(1-x)^{\frac{m-q}{2}}(1+x)^{\frac{m+q}{2}}P_{l-m}^{(m-q,m+q)}(x), \tag{2}$$

for $-l \leqslant m \leqslant +l$, and apply the Jacobi differential equation, then we shall get the following differential equation:

$$(1-x^2)\mathcal{P}_{l,m}''^{(q)}(x) - 2x\mathcal{P}_{l,m}'^{(q)}(x) + \left(l(l+1) - \frac{q^2+m^2-2mqx}{1-x^2}\right)\mathcal{P}_{l,m}^{(q)}(x) = 0. \tag{3}$$

The differential equation (3) is invariant under the exchange of positions of m and q parameters. This implies that $\mathcal{P}_{l,q}^{(m)}(x)$ is another solution to equation (3). Furthermore, since equation (3) is unaltered when m and q are replaced by $-m$ and $-q$, respectively, the function $\mathcal{P}_{l,-m}^{(-q)}(x)$ is also another solution to it. We now note the following useful relation that follows immediately from the Leibnitz rule:

$$(-1)^{q+m}\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{l+m}((1-x)^{l+q}(1+x)^{l-q})$$
$$=\frac{\Gamma(l+m+1)}{\Gamma(l-m+1)}(1-x)^{q-m}(1+x)^{-q-m}\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{l-m}((1-x)^{l-q}(1+x)^{l+q}). \tag{4}$$

Thus, for the differential equation (3), the solutions $\mathcal{P}^{(-q)}_{l,-m}(x)$ can also be considered as

$$\mathcal{P}^{(-q)}_{l,-m}(x)=\frac{a_{l,-m}(-q)}{a_{l,m}(q)}\frac{\Gamma(l+m+1)}{(-1)^{m+q}\Gamma(l-m+1)}\mathcal{P}^{(q)}_{l,m}(x). \tag{5}$$

It allows us not only to recover all bound states of the monopole problem but also to obtain positive and negative discrete representations of the deformed $su(1,1)$ algebra via monopole harmonics as an internal symmetry. Clearly, if we demand that the following differential operators [13]

$$\mathcal{A}_{\pm}(m;q;x)=\pm\sqrt{1-x^2}\frac{\mathrm{d}}{\mathrm{d}x}+\frac{\left(m-\frac{1}{2}\mp\frac{1}{2}\right)x-q}{\sqrt{1-x^2}}, \tag{6a}$$

$$\mathcal{A}_{\pm}(l,m;q;x)=\pm(1-x^2)\frac{\mathrm{d}}{\mathrm{d}x}-lx+q\frac{m}{l} \tag{6b}$$

simultaneously satisfy the following raising and lowering relations with respect to both indices l and m,

$$\mathcal{A}_{+}(m;q;x)\mathcal{P}^{(q)}_{l,m-1}(x)=\sqrt{(l-m+1)(l+m)}\mathcal{P}^{(q)}_{l,m}(x), \tag{7a}$$

$$\mathcal{A}_{-}(m;q;x)\mathcal{P}^{(q)}_{l,m}(x)=\sqrt{(l-m+1)(l+m)}\mathcal{P}^{(q)}_{l,m-1}(x), \tag{7b}$$

$$\mathcal{A}_{+}(l,m;q;x)\mathcal{P}^{(q)}_{l-1,m}(x)=\sqrt{\frac{(l^2-m^2)(l^2-q^2)(2l-1)}{l^2(2l+1)}}\mathcal{P}^{(q)}_{l,m}(x), \tag{8a}$$

$$\mathcal{A}_{-}(l,m;q;x)\mathcal{P}^{(q)}_{l,m}(x)=\sqrt{\frac{(l^2-m^2)(l^2-q^2)(2l+1)}{l^2(2l-1)}}\mathcal{P}^{(q)}_{l-1,m}(x), \tag{8b}$$

so that the following orthonormality relation is also realized:

$$\int_{-1}^{1}\mathcal{P}^{(q)}_{l,m}(x)\mathcal{P}^{(q)}_{l',m}(x)\,\mathrm{d}x=\delta_{ll'}, \tag{9}$$

then the normalization coefficients $a_{l,m}(q)$ are calculated as

$$a_{l,m}(q)=\frac{1}{(-1)^m2^l}\sqrt{\frac{\Gamma(l+m+1)}{\Gamma(l-m+1)\Gamma(l+q+1)\Gamma(l-q+1)}}. \tag{10}$$

Note that in contrast to $0\leqslant m\leqslant n$ in [13], $-l\leqslant m\leqslant +l$ is a symmetric interval. Substituting equation (10) into equation (2), we obtain the following two symmetric properties for all positive and negative values of m:

$$\mathcal{P}^{(q)}_{l,m}(x)=(-1)^{m+q}\mathcal{P}^{(-q)}_{l,-m}(x), \tag{11a}$$

$$\mathcal{P}^{(q)}_{l,m}(x)=(-1)^{q-m}\mathcal{P}^{(m)}_{l,q}(x). \tag{11b}$$

If we use equation (11a), it becomes obvious that equations (7a) and (7b) are transformed to lowering and raising relations corresponding to the index $-m$ of $\mathcal{P}_{l,-m}^{(-q)}(x)$ functions, but equations (8a) and (8b) remain the same as raising and lowering relations of index l of those functions respectively.

The surface of a sphere is usually covered by two coordinate patches as the open neighborhoods of north and south poles as $R_N = \{0 \leqslant \theta < \pi\}$ and $R_S = \{0 < \theta \leqslant \pi\}$, named north and south hemispheres, respectively. The second coordinate for both patches is the auxiliary variable $0 \leqslant \phi < 2\pi$. The monopole harmonics $Y_{lm}^{q}(\theta,\phi)$ as the sections of the fiber bundle are defined by two local components $Y_{lm}^{N;q}(\theta,\phi)$ and $Y_{lm}^{S;q}(\theta,\phi)$ on the coordinate patches R_N and R_S separately, so that the second component is equal to the product of the phase factor as $\mathrm{e}^{-2\mathrm{i}q\phi}$ and the first one:

$$Y_{lm}^{N(S);q}(\theta,\phi) := \frac{\mathrm{e}^{\mathrm{i}(m\pm q)\phi}}{\sqrt{2\pi}}\mathcal{P}_{l,m}^{(q)}(-\cos\theta). \tag{12}$$

Now, it is necessary to emphasize that the monopole harmonics with $0 \leqslant m \leqslant l$ have two different forms:

$$\begin{aligned}Y_{lm}^{N(S);q}(\theta,\phi) &= \frac{\mathrm{e}^{\mathrm{i}(m\pm q)\phi}}{(-1)^m 2^l}\sqrt{\frac{(2l+1)\Gamma(l+m+1)}{4\pi\Gamma(l-m+1)\Gamma(l+q+1)\Gamma(l-q+1)}}\\ &\quad\times\frac{(1+\cos\theta)^{\frac{q-m}{2}}}{(1-\cos\theta)^{\frac{q+m}{2}}}\left(\frac{1}{\sin\theta}\frac{\mathrm{d}}{\mathrm{d}\theta}\right)^{l-m}[(1+\cos\theta)^{l-q}(1-\cos\theta)^{l+q}],\end{aligned} \tag{13a}$$

$$\begin{aligned}Y_{l-m}^{N(S);q}(\theta,\phi) &= \frac{\mathrm{e}^{\mathrm{i}(-m\pm q)\phi}(-1)^q}{2^l}\sqrt{\frac{(2l+1)\Gamma(l+m+1)}{4\pi\Gamma(l-m+1)\Gamma(l+q+1)\Gamma(l-q+1)}}\\ &\quad\times\frac{(1-\cos\theta)^{\frac{q-m}{2}}}{(1+\cos\theta)^{\frac{q+m}{2}}}\left(\frac{1}{\sin\theta}\frac{\mathrm{d}}{\mathrm{d}\theta}\right)^{l-m}[(1+\cos\theta)^{l+q}(1-\cos\theta)^{l-q}].\end{aligned} \tag{13b}$$

From (11a), we get

$$Y_{lm}^{*N(S);q}(\theta,\phi) = (-1)^{m+q}Y_{l-m}^{N(S);-q}(\theta,\phi), \tag{14}$$

which is immediately verified by (13a) and (13b). One can show that monopole harmonics $Y_{lm}^{N(S);q}(\theta,\phi)$ for positive and negative values of m, i.e. $-l \leqslant m \leqslant l$, satisfy the following differential equation:

$$\begin{aligned}&\left[\frac{\partial^2}{\partial\theta^2} + \cot\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2} + 2\mathrm{i}q\frac{\cos\theta\mp 1}{\sin^2\theta}\frac{\partial}{\partial\varphi}\right.\\ &\qquad\left.\pm 2q^2\frac{\cos\theta\mp 1}{\sin^2\theta} + l(l+1)\right]Y_{lm}^{N(S);q}(\theta,\phi) = 0.\end{aligned} \tag{15}$$

With the help of the orthonormality relation of the $\mathcal{P}_{l,m}^{(q)}(x)$ functions given in (9), it is straightforward to conclude that the monopole harmonics for a fixed magnetic charge q form a complete set of sections in the angular space

$$\int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi} Y_{lm}^{*N(S);q}(\theta,\phi)Y_{l'm'}^{N(S);q}(\theta,\phi)\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi = \delta_{ll'}\delta_{mm'}. \tag{16}$$

Therefore, we can introduce an infinite-dimensional Hilbert space of monopole harmonics: $\mathcal{H}^{N(S)} := \mathrm{span}\{Y_{lm}^{N(S);q}(\theta,\phi), l \geqslant 0, -l \leqslant m \leqslant l\}$.

3. Simultaneous realization of two different types of laddering equations by the monopole harmonics $Y_{lm}^{N(S);q}(\theta,\phi)$

In [13], the ladder symmetry with respect to the azimuthal quantum number l has only been formulated for the positive quantum numbers m as $0 \leqslant m \leqslant n$. In order to realize the positive and negative integer discrete representations of the deformed $su(1,1)$ algebra we have extended the ladder symmetry to $-l \leqslant m \leqslant l$ here. It is well known that the monopole harmonics represent the $su(2)$ Lie algebra as

$$L_+^{N(S);q} Y_{lm-1}^{N(S);q}(\theta,\phi) = \sqrt{(l+m)(l-m+1)}\, Y_{lm}^{N(S);q}(\theta,\phi), \tag{17a}$$

$$L_-^{N(S);q} Y_{lm}^{N(S);q}(\theta,\phi) = \sqrt{(l+m)(l-m+1)}\, Y_{lm-1}^{N(S);q}(\theta,\phi), \tag{17b}$$

$$L_3^{N(S);q} Y_{lm}^{N(S);q}(\theta,\phi) = m Y_{lm}^{N(S);q}(\theta,\phi), \tag{17c}$$

with the following explicit differential forms for the operators:

$$L_+^{N(S);q} = \mathrm{e}^{\mathrm{i}\phi}\left(\frac{\partial}{\partial\theta} + \mathrm{i}\cot\theta\frac{\partial}{\partial\phi} \pm q\cot\theta - \frac{q}{\sin\theta}\right), \tag{18a}$$

$$L_-^{N(S);q} = \mathrm{e}^{-\mathrm{i}\phi}\left(-\frac{\partial}{\partial\theta} + \mathrm{i}\cot\theta\frac{\partial}{\partial\phi} \pm q\cot\theta - \frac{q}{\sin\theta}\right), \tag{18b}$$

$$L_3^{N(S);q} = -\mathrm{i}\frac{\partial}{\partial\phi} \mp q. \tag{18c}$$

Relations (17a) and (17b) immediately follow from (7a) and (7b) for both intervals $0 \leqslant m \leqslant l$ and $-l \leqslant m \leqslant 0$, respectively. The operators $L_+^{N(S);q}$, $L_-^{N(S);q}$ and $L_3^{N(S);q}$ satisfy the $su(2)$ commutation relations as

$$\left[L_+^{N(S);q}, L_-^{N(S);q}\right] = 2L_3^{N(S);q}, \qquad \left[L_3^{N(S);q}, L_\pm^{N(S);q}\right] = \pm L_\pm^{N(S);q}. \tag{19}$$

The highest and lowest states $Y_{ll}^{N(S);q}(\theta,\phi)$ and $Y_{l-l}^{N(S);q}(\theta,\phi)$ are annihilated by the operators $L_+^{N(S);q}$ and $L_-^{N(S);q}$ respectively as

$$L_\pm^{N(S);q} Y_{l\pm l}^{N(S);q}(\theta,\phi) = 0. \tag{20}$$

The solutions of these first-order differential equations are

$$Y_{ll}^{N(S);q}(\theta,\phi) = \frac{1}{(-2)^l}\sqrt{\frac{\Gamma(2l+2)}{4\pi\Gamma(l+q+1)\Gamma(l-q+1)}}\,\mathrm{e}^{\mathrm{i}(l\pm q)\phi}\frac{(1-\cos\theta)^{\frac{q+l}{2}}}{(1+\cos\theta)^{\frac{q-l}{2}}}, \tag{21a}$$

$$Y_{l-l}^{N(S);q}(\theta,\phi) = \frac{(-1)^q}{2^l}\sqrt{\frac{\Gamma(2l+2)}{4\pi\Gamma(l+q+1)\Gamma(l-q+1)}}\,\mathrm{e}^{-\mathrm{i}(l\pm q)\phi}\frac{(1+\cos\theta)^{\frac{q+l}{2}}}{(1-\cos\theta)^{\frac{q-l}{2}}}. \tag{21b}$$

The monopole harmonics are produced by the repetitive application of both raising and lowering relations of the index m given in (17a) and (17b), as follows:

$$Y_{l\pm m}^{N(S);q}(\theta,\phi) = \sqrt{\frac{\Gamma(l+m+1)}{\Gamma(l-m+1)\Gamma(2l+1)}}\left[L_\mp^{N(S);q}\right]^{l-m} Y_{l\pm l}^{N(S);q}(\theta,\phi), \tag{22}$$

where $-l \leqslant m \leqslant l$. It should be emphasized that in each of the relations of (22), the magnetic charge q can be replaced by $-q$. Therefore, the commutation relations (19) not only quantize the index m, but also such a realization of $su(2)$ algebra is responsible for the generation of monopole harmonics. Remarking relation ($11b$), the commutation relations (19) are also responsible for quantization of the magnetic monopole charge q. The Casimir operator corresponding to this algebra, i.e.

$$(\mathbf{L}^{N(S);q})^2 = L_+^{N(S);q} L_-^{N(S);q} + \left(L_3^{N(S);q}\right)^2 - L_3^{N(S);q} = -\Delta + 2q\frac{\cos\theta \mp 1}{\sin^2\theta} L_3^{N(S);q}, \tag{23}$$

where the Laplace operator Δ is defined as

$$\Delta = \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}, \tag{24}$$

satisfies the following eigenvalue equations on the space of monopole harmonics

$$(\mathbf{L}^{N(S);q})^2 Y_{lm}^{N(S);q}(\theta,\phi) = l(l+1) Y_{lm}^{N(S);q}(\theta,\phi) \qquad -l \leqslant m \leqslant l. \tag{25}$$

Consequently, the finite-dimensional Hilbert subspaces $\mathcal{H}^{N(S);l} := \operatorname{span}\left\{Y_{lm}^{N(S);q}(\theta,\phi)\right\}_{m=-l}^{l}$ constitute irreducible representations for $su(2)$ via the magnetic quantum number m.

We use equations ($6b$), ($8a$) and ($8b$) to obtain the ladder operators which shift the azimuthal quantum number l of monopole harmonics, i.e.

$$J_+^{N(S);q}(l) = \sin\theta\frac{\partial}{\partial\theta} - \mathrm{i}\frac{q}{l}\frac{\partial}{\partial\phi} + l\cos\theta \mp \frac{q^2}{l}, \tag{26a}$$

$$J_-^{N(S);q}(l) = -\sin\theta\frac{\partial}{\partial\theta} - \mathrm{i}\frac{q}{l}\frac{\partial}{\partial\phi} + l\cos\theta \mp \frac{q^2}{l}, \tag{26b}$$

and the corresponding laddering relations for $-l \leqslant m \leqslant l$:

$$J_+^{N(S);q}(l) Y_{l-1m}^{N(S);q}(\theta,\phi) = \sqrt{\frac{(l^2-m^2)(l^2-q^2)(2l-1)}{l^2(2l+1)}}\, Y_{lm}^{N(S);q}(\theta,\phi), \tag{27a}$$

$$J_-^{N(S);q}(l) Y_{lm}^{N(S);q}(\theta,\phi) = \sqrt{\frac{(l^2-m^2)(l^2-q^2)(2l+1)}{l^2(2l-1)}}\, Y_{l-1m}^{N(S);q}(\theta,\phi). \tag{27b}$$

It is easy to see that the monopole harmonics $Y_{m\pm m}^{N(S);q}(\theta,\phi)$ with $m > 0$ as the lowest states are annihilated by the lowering operators $J_-^{N(S);q}(m)$:

$$J_-^{N(S);q}(m) Y_{mm}^{N(S);q}(\theta,\phi) = 0, \tag{28a}$$

$$J_-^{N(S);q}(m) Y_{m-m}^{N(S);q}(\theta,\phi) = 0. \tag{28b}$$

The solution of the first-order differential equation (28a) is ($21a$), and the solution of ($28b$) is ($21b$) if l is replaced by m. For a given m, by repeated application of the raising relation ($27a$), one may get the arbitrary monopole harmonics from $Y_{m\pm m}^{N(S);q}(\theta,\phi)$:

$$\begin{aligned} Y_{l\pm m}^{N(S);q}(\theta,\phi) = {} & \frac{\Gamma(l+1)}{\Gamma(m+1)}\sqrt{\frac{(2l+1)\Gamma(2m+1)\Gamma(m+q+1)\Gamma(m-q+1)}{(2m+1)\Gamma(l+m+1)\Gamma(l-m+1)\Gamma(l+q+1)\Gamma(l-q+1)}} \\ & \times J_+^{N(S);q}(l) J_+^{N(S);q}(l-1)\cdots J_+^{N(S);q}(m+1) Y_{m\pm m}^{N(S);q}(\theta,\phi). \end{aligned} \tag{29}$$

In this section, the ladder symmetry was extended to the monopole harmonics with negative magnetic quantum number, just similar to the realization of the $su(2)$ Lie algebra symmetry which has been known before for $-l \leqslant m \leqslant l$. Therefore, the infinite-dimensional Hilbert subspaces $\mathcal{H}^{N(S);m} := \mathrm{span}\{Y_{lm}^{N(S);q}(\theta,\phi)\}_{l=m}^{\infty}$ represent the ladder symmetry with respect to the azimuthal quantum number l, irreducibly. From an algebraic point of view, equation (29) is related to the ladder symmetry which serves as the formal spectrum-generating algebra. The approach of shifting the azimuthal quantum number l, i.e. (29), is different from $su(2)$ irreducible representation approach for shifting the magnetic quantum number m, given in (22).

4. Monopole harmonics and irreducible discrete representations of the deformed $su(1,1)$ algebra

Let us first define

$$K_{\pm\pm}^{N;q}(l) := \pm\left[L_{\pm}^{N;q}, J_{\pm}^{N;q}(l)\right] = \mathrm{e}^{\pm\mathrm{i}\phi}\left[\mp\left(\frac{q}{l}-\cos\theta\right)\frac{\partial}{\partial\theta} - \mathrm{i}\left(\left(\frac{q}{l}-\cos\theta\right)\cot\theta - \sin\theta\right)\frac{\partial}{\partial\phi} + (q-l)\sin\theta + q\left(\frac{q}{l}-\cos\theta\right)\frac{1-\cos\theta}{\sin\theta}\right] \tag{30a}$$

$$K_{\pm\pm}^{S;q}(l) := \pm\left[L_{\pm}^{S;q}, J_{\pm}^{S;q}(l)\right] = \mathrm{e}^{\pm\mathrm{i}\phi}\left[\mp\left(\frac{q}{l}-\cos\theta\right)\frac{\partial}{\partial\theta} - \mathrm{i}\left(\left(\frac{q}{l}-\cos\theta\right)\cot\theta - \sin\theta\right)\frac{\partial}{\partial\phi} - (q+l)\sin\theta + q\left(\frac{q}{l}-\cos\theta\right)\frac{1+\cos\theta}{\sin\theta}\right] \tag{30b}$$

$$K_{\pm\mp}^{N;q}(l) := \pm\left[L_{\mp}^{N;q}, J_{\pm}^{N;q}(l)\right] = \mathrm{e}^{\mp\mathrm{i}\phi}\left[\mp\left(\frac{q}{l}+\cos\theta\right)\frac{\partial}{\partial\theta} + \mathrm{i}\left(\left(\frac{q}{l}+\cos\theta\right)\cot\theta + \sin\theta\right)\frac{\partial}{\partial\phi} + (q+l)\sin\theta - q\left(\frac{q}{l}+\cos\theta\right)\frac{1-\cos\theta}{\sin\theta}\right] \tag{30c}$$

$$K_{\pm\mp}^{S;q}(l) := \pm\left[L_{\mp}^{S;q}, J_{\pm}^{S;q}(l)\right] = \mathrm{e}^{\mp\mathrm{i}\phi}\left[\mp\left(\frac{q}{l}+\cos\theta\right)\frac{\partial}{\partial\theta} + \mathrm{i}\left(\left(\frac{q}{l}+\cos\theta\right)\cot\theta + \sin\theta\right)\frac{\partial}{\partial\phi} + (l-q)\sin\theta - q\left(\frac{q}{l}+\cos\theta\right)\frac{1+\cos\theta}{\sin\theta}\right]. \tag{30d}$$

Using equations (17*a*), (17*b*), (27*a*) and (27*b*), it is easily shown that

$$K_{++}^{N(S);q}(l)Y_{l-1m-1}^{N(S);q}(\theta,\phi) = \sqrt{\frac{(2l-1)(l+m)(l+m-1)(l^2-q^2)}{l^2(2l+1)}}Y_{lm}^{N(S);q}(\theta,\phi), \tag{31a}$$

$$K_{--}^{N(S);q}(l)Y_{lm}^{N(S);q}(\theta,\phi) = \sqrt{\frac{(2l+1)(l+m)(l+m-1)(l^2-q^2)}{l^2(2l-1)}}Y_{l-1m-1}^{N(S);q}(\theta,\phi), \tag{31b}$$

$$K_{+-}^{N(S);q}(l)Y_{l-1m+1}^{N(S);q}(\theta,\phi) = \sqrt{\frac{(2l-1)(l-m)(l-m-1)(l^2-q^2)}{l^2(2l+1)}}Y_{lm}^{N(S);q}(\theta,\phi), \tag{31c}$$

$$K_{-+}^{N(S);q}(l)Y_{lm}^{N(S);q}(\theta,\phi) = \sqrt{\frac{(2l+1)(l-m)(l-m-1)(l^2-q^2)}{l^2(2l-1)}}Y_{l-1m+1}^{N(S);q}(\theta,\phi). \tag{31d}$$

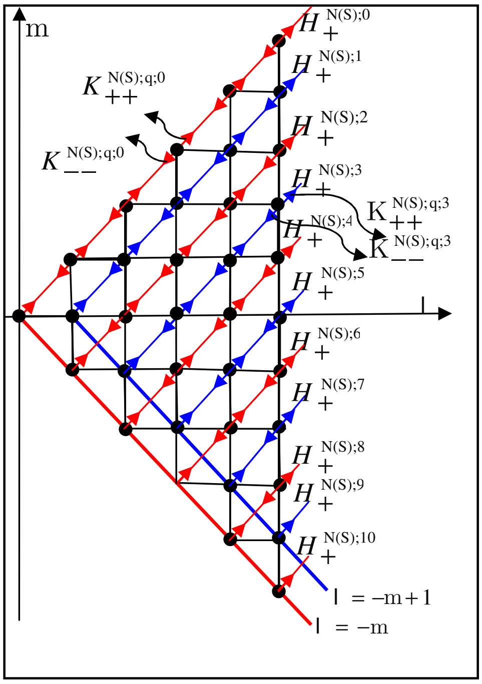

Figure 1. The monopole harmonics lattice corresponding to the bases of the integer positive irreducible representation subspaces $\mathcal{H}_+^{N(S);d}$ for given values of $l - m$.

Thus, the laddering equations (17*a*), (17*b*), (27*a*) and (27*b*), which shift m and l with the restrictions $-l \leqslant m \leqslant l$ and $l \geqslant 0$ respectively, lead to the derivation of two different types of laddering relations for the monopole harmonics $Y_{lm}^{N(S);q}(\theta, \phi)$. They are realized by two pairs of ladder operators whose corresponding laddering equations shift both of the indices l and m simultaneously and agreeably, and simultaneously and inversely, respectively. Now we are in a position that for appropriate configurations of the monopole harmonics, we can realize the positive and negative integer discrete representations of a deformed $su(1, 1)$ algebra. It describes an internal additional symmetry for the magnetic monopole problem via surface bound states.

4.1. Positive integer irreducible discrete representations of the deformed $su(1, 1)$ algebra for $l - m$

Let us relabel the monopole harmonics by a new parameter $d = l - m$ with $d = 0, 1, 2, \ldots,$ instead of l: $Y_{d+mm}^{N(S);q}(\theta, \phi)$. Consequently, the Hilbert space $\mathcal{H}^{N(S)}$ can be split into infinite direct sums of infinite-dimensional Hilbert subspaces as $\mathcal{H}^{N(S)} = \oplus_{d=0}^{\infty}\mathcal{H}_+^{N(S);d}$. In figure 1, we have schematically shown all bases of the Hilbert space $\mathcal{H}^{N(S)}$ as the points (l, m) with the $-l \leqslant m \leqslant l$ limitation in the flat plane with l and m as the horizontal and vertical axes, respectively. It is clear that the Hilbert subspaces $\mathcal{H}_+^{N(S);d}$ are seen as inclined lines along the bisector of the first quadrant of the plane. They are classified into two different classes: $\mathcal{H}_+^{N(S);d=2k} = \text{span}\{Y_{2k+mm}^{N(S);q}(\theta, \phi)\}_{m=-k}^{\infty}$ and $\mathcal{H}_+^{N(S);d=2k+1} = \text{span}\{Y_{2k+m+1m}^{N(S);q}(\theta, \phi)\}_{m=-k}^{\infty}$ with $k = 0, 1, 2, \ldots$. It is not difficult to see from (31*a*) and (31*b*) that the following relations on

$\mathcal{H}_+^{N(S);d}$ should be satisfied:

$$\mathcal{K}_{++}^{N(S);q;d} Y_{d+m-1m-1}^{N(S);q}(\theta,\phi) = \sqrt{\frac{(2d+2m-1)(d+2m)(d+2m-1)(d+m+q)(d+m-q)}{2d+2m+1}} \times Y_{d+mm}^{N(S);q}(\theta,\phi), \tag{32a}$$

$$\mathcal{K}_{--}^{N(S);q;d} Y_{d+mm}^{N(S);q}(\theta,\phi) = \sqrt{\frac{(2d+2m+1)(d+2m)(d+2m-1)(d+m+q)(d+m-q)}{2d+2m-1}} \times Y_{d+m-1m-1}^{N(S);q}(\theta,\phi), \tag{32b}$$

$$\mathcal{K}_3^{N(S);q} Y_{d+mm}^{N(S);q}(\theta,\phi) = m Y_{d+mm}^{N(S);q}(\theta,\phi). \tag{32c}$$

The explicit differential forms of the above operators are

$$\begin{aligned}\mathcal{K}_{\pm\pm}^{N;q;d} = \mathrm{e}^{\pm\mathrm{i}\phi}\Bigg[&\mp\left(q-\left(d-q+\frac{1}{2}\pm\frac{1}{2}\right)\cos\theta\right)\frac{\partial}{\partial\theta} \mp \mathrm{i}\cos\theta\frac{\partial^2}{\partial\theta\partial\phi} + \left(\frac{1}{\sin\theta}+\sin\theta\right)\frac{\partial^2}{\partial\phi^2}\\ &+\mathrm{i}\left(2\left(d-q+\frac{1}{2}\pm\frac{1}{2}\right)\sin\theta - \frac{2q-d-\frac{1}{2}\mp\frac{1}{2}}{\sin\theta}\right)\frac{\partial}{\partial\phi}\\ &+q\frac{q-\left(d-q+\frac{1}{2}\pm\frac{1}{2}\right)\cos\theta}{\sin\theta}(1-\cos\theta)\\ &-\left(d-q+\frac{1}{2}\pm\frac{1}{2}\right)\left(d-2q+\frac{1}{2}\pm\frac{1}{2}\right)\sin\theta\Bigg],\end{aligned} \tag{33a}$$

$$\begin{aligned}\mathcal{K}_{\pm\pm}^{S;q;d} = \mathrm{e}^{\pm\mathrm{i}\phi}\Bigg[&\mp\left(q-\left(d+q+\frac{1}{2}\pm\frac{1}{2}\right)\cos\theta\right)\frac{\partial}{\partial\theta} \mp \mathrm{i}\cos\theta\frac{\partial^2}{\partial\theta\partial\phi} + \left(\frac{1}{\sin\theta}+\sin\theta\right)\frac{\partial^2}{\partial\phi^2}\\ &+\mathrm{i}\left(2\left(d+q+\frac{1}{2}\pm\frac{1}{2}\right)\sin\theta + \frac{2q+d+\frac{1}{2}\pm\frac{1}{2}}{\sin\theta}\right)\frac{\partial}{\partial\phi}\\ &+q\frac{q-\left(d+q+\frac{1}{2}\pm\frac{1}{2}\right)\cos\theta}{\sin\theta}(1+\cos\theta)\\ &-\left(d+q+\frac{1}{2}\pm\frac{1}{2}\right)\left(d+2q+\frac{1}{2}\pm\frac{1}{2}\right)\sin\theta\Bigg],\end{aligned} \tag{33b}$$

$$\mathcal{K}_3^{N(S);q} = -\mathrm{i}\frac{\partial}{\partial\phi} \mp q. \tag{33c}$$

Note that before substituting $d+m$ instead of l in equations (31a) and (31b), we have multiplied both sides by l. Also, equation (32c) is directly followed by (33c).

The operators $\mathcal{K}_{\pm\pm}^{N(S);q;d}$, similar to the angular momentum operators $L_{\pm}^{N(S);q}$, are linear in terms of $\frac{\partial}{\partial\theta}$. However, the commutator of the two operators $\mathcal{K}_{++}^{N(S);q;d}$ and $\mathcal{K}_{--}^{N(S);q;d}$ is calculated as a quadratic expression in terms of $\frac{\partial}{\partial\theta}$. Since it is known that these operators should just act on monopole harmonics belonging to the Hilbert subspace $\mathcal{H}_+^{N(S);d}$, we take into account the following operator equality:

$$\begin{aligned}\frac{\partial^2}{\partial\theta^2} = &-\cot\theta\frac{\partial}{\partial\theta} - \cot^2\theta\frac{\partial^2}{\partial\phi^2} - \mathrm{i}\left(\frac{2q(\cos\theta\mp1)}{\sin^2\theta} - 2d \pm 2q - 1\right)\frac{\partial}{\partial\phi} + 2q^2\frac{1\mp\cos\theta}{\sin^2\theta}\\ &-(d\mp q)(d\mp q+1).\end{aligned} \tag{34}$$

We can easily use the operator equality (34) to check that the functionality in terms of θ is omitted in the commutation relation $\left[\mathcal{K}_{++}^{N(S);q;d}, \mathcal{K}_{--}^{N(S);q;d}\right]$:

$$\begin{aligned}
\left[\mathcal{K}_{++}^{N(S);q;d}, \mathcal{K}_{--}^{N(S);q;d}\right] &= -16\left(\mathcal{K}_3^{N(S);q}\right)^3 - 18(2d+1)\left(\mathcal{K}_3^{N(S);q}\right)^2 - 2(13d(d+1) \\
&\quad - 4q^2 + 5)\mathcal{K}_3^{N(S);q} - (3d^2(2d+3) + d(7-4q^2) + 2(1-q^2)), \\
\left[\mathcal{K}_3^{N(S);q}, \mathcal{K}_{++}^{N(S);q;d}\right] &= +\mathcal{K}_{++}^{N(S);q;d}, \\
\left[\mathcal{K}_3^{N(S);q}, \mathcal{K}_{--}^{N(S);q;d}\right] &= -\mathcal{K}_{--}^{N(S);q;d}.
\end{aligned} \tag{35}$$

The second and third relations of (35) are directly followed without using (34). The Casimir operator corresponding to this nonlinear algebra can be calculated to give (see, for example, [14, 15])

$$\begin{aligned}
\mathcal{C}^{N(S);q;d} &= \mathcal{K}_{++}^{N(S);q;d}\mathcal{K}_{--}^{N(S);q;d} - 4\left(\mathcal{K}_3^{N(S);q}\right)^4 + 2(1-6d)\left(\mathcal{K}_3^{N(S);q}\right)^3 \\
&\quad + (4q^2 - 13d^2 + 5d)\left(\mathcal{K}_3^{N(S);q}\right)^2 + (-6d^3 + 4d(d+q^2) - 2q^2)\mathcal{K}_3^{N(S);q}.
\end{aligned} \tag{36}$$

It satisfies the following eigenvalue equations:

$$\mathcal{C}^{N(S);q;d} Y_{d+mm}^{N(S);q}(\theta, \phi) = d(d-1)(d+q)(d-q) Y_{d+mm}^{N(S);q}(\theta, \phi) \tag{37}$$

on the Hilbert subspace $\mathcal{H}_+^{N(S);d}$. Therefore, the nonlinear commutation relations (35) present a deformed algebra of $su(1,1)$ with the $(l-m)$-integer positive irreducible representation given in relations (32).

The nonlinear commutation relations (35) constitute a spectrum-generating algebra since they in turn lead to an algebraic method for deriving monopole harmonics belonging to $\mathcal{H}_+^{N(S);d}$. From equation (32b) it becomes obvious that the monopole harmonics $Y_{k-k}^{N(S);q}(\theta, \phi)$ and $Y_{k+1-k}^{N(S);q}(\theta, \phi)$ are the lowest states of the Hilbert subspaces $\mathcal{H}_+^{N(S);2k}$ and $\mathcal{H}_+^{N(S);2k+1}$, respectively:

$$\mathcal{K}_{--}^{N(S);q;2k} Y_{k-k}^{N(S);q}(\theta, \phi) = 0, \tag{38a}$$

$$\mathcal{K}_{--}^{N(S);q;2k+1} Y_{k+1-k}^{N(S);q}(\theta, \phi) = 0. \tag{38b}$$

Equations (38a) and (38b) are first-order differential equations and have the following solutions:

$$Y_{k-k}^{N(S);q}(\theta, \phi) = \frac{(-1)^k\, \mathrm{e}^{\mathrm{i}(-k\pm q)\phi}}{2^{k+1}\sqrt{\pi}} \sqrt{\frac{\Gamma(2k+2)}{\Gamma(k+q+1)\Gamma(k-q+1)}} (1-\cos\theta)^{\frac{k-q}{2}} (1+\cos\theta)^{\frac{k+q}{2}}, \tag{39a}$$

$$\begin{aligned}
Y_{k+1-k}^{N(S);q}(\theta, \phi) &= \frac{(-1)^q\, \mathrm{e}^{\mathrm{i}(-k\pm q)\phi}}{2^{k+\frac{1}{2}}\sqrt{\pi}} \sqrt{\frac{(2k+3)\Gamma(2k+2)}{\Gamma(k+q+2)\Gamma(k-q+2)}} (1-\cos\theta)^{\frac{k-q}{2}} (1+\cos\theta)^{\frac{k+q}{2}} \\
&\quad \times ((k+1)\cos\theta - q).
\end{aligned} \tag{39b}$$

The remaining monopole harmonics belonging to the representation spaces $\mathcal{H}_+^{N(S);2k}$ and $\mathcal{H}_+^{N(S);2k+1}$ can be calculated by successive use of the raising relation (32a), that is

$$\begin{aligned}
Y_{m+2km}^{N(S);q}(\theta, \phi) &= \sqrt{\frac{(2m+4k+1)\Gamma(k+q+1)\Gamma(k-q+1)}{(2k+1)\Gamma(m+2k+q+1)\Gamma(m+2k-q+1)\Gamma(2m+2k+1)}} \\
&\quad \times \left[\mathcal{K}_{++}^{N(S);q;2k}\right]^{m+k} Y_{k-k}^{N(S);q}(\theta, \phi),
\end{aligned} \tag{40a}$$

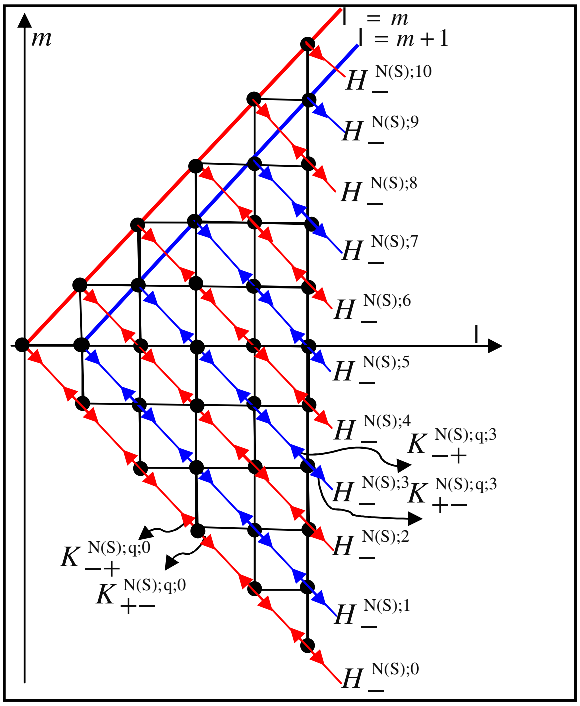

Figure 2. The monopole harmonics lattice corresponding to the bases of the integer negative irreducible representation subspaces $\mathcal{H}_-^{N(S);d'}$ for given values of $l+m$.

$$Y^{N(S);q}_{m+2k+1m}(\theta,\phi)=\sqrt{\frac{(2m+4k+3)\Gamma(k+q+2)\Gamma(k-q+2)}{(2k+3)\Gamma(m+2k+q+2)\Gamma(m+2k-q+2)\Gamma(2k+2m+2)}}$$
$$\times\left[\mathcal{K}_{++}^{N(S);q;2k+1}\right]^{m+k}Y^{N(S);q}_{k+1-k}(\theta,\phi), \qquad (40b)$$

where $m=-k,-k+1,-k+2,\ldots.$ While calculating explicit forms of the monopole harmonics, the expression $\frac{\partial^2}{\partial\theta^2}$ appears on the right hand side of relations (40a) and (40b) the same as every other one. Therefore, one can alternatively use the operator equality (34).

4.2. *Negative integer irreducible discrete representation of the deformed* $su(1,1)$ *algebra for* $l+m$

If relations (31c) and (31d) are used as successive shifts, then they can be considered as laddering relations with respect to one parameter. In other words by the definition $d'=l+m$ with $d'=0,1,2,\ldots,$ the Hilbert space $\mathcal{H}^{N(S)}$ may be taken into account as the direct sum of the disjoint infinite-dimensional Hilbert subspaces: $\mathcal{H}^{N(S)}=\oplus_{d'=0}^{\infty}\mathcal{H}_-^{N(S);d'}$. Figure 2 shows decompositions of the Hilbert space $\mathcal{H}^{N(S)}$ into distinct classes of the Hilbert subspaces made in two different ways, which are also different from those in figure 1: $\mathcal{H}_-^{N(S);d'=2s}=\mathrm{span}\{Y^{N(S);q}_{2s-mm}(\theta,\phi)\}_{m=s}^{-\infty}$ and $\mathcal{H}_-^{N(S);d'=2s+1}=\mathrm{span}\{Y^{N(S);q}_{2s-m+1m}(\theta,\phi)\}_{m=s}^{-\infty}$ with $s=0,1,2,\ldots.$ The inclined lines along the bisector of the second quadrant of the plane indicate these two types of decomposition of the Hilbert space schematically. Before we again substitute $d' - m$ for l in equations (31c) and (31d), we multiply them by l. Then, one can conclude that the monopole harmonics belonging to $\mathcal{H}_-^{N(S);d'}$ satisfy the following relations:

$$\begin{aligned}
\mathcal{K}_{+-}^{N(S);q;d'} Y_{d'-m-1m+1}^{N(S);q}(\theta,\phi) &= \sqrt{\frac{(2d'-2m-1)(d'-2m)(d'-2m-1)(d'-m+q)(d'-m-q)}{2d'-2m+1}} \\
&\quad \times Y_{d'-mm}^{N(S);q}(\theta,\phi),
\end{aligned} \tag{41a}$$

$$\begin{aligned}
\mathcal{K}_{-+}^{N(S);q;d'} Y_{d'-mm}^{N(S);q}(\theta,\phi) &= \sqrt{\frac{(2d'-2m+1)(d'-2m)(d'-2m-1)(d'-m+q)(d'-m-q)}{2d'-2m-1}} \\
&\quad \times Y_{d'-m-1m+1}^{N(S);q}(\theta,\phi),
\end{aligned} \tag{41b}$$

$$\mathcal{K}_3^{N(S);q} Y_{d'-mm}^{N(S);q}(\theta,\phi) = m Y_{d'-mm}^{N(S);q}(\theta,\phi), \tag{41c}$$

where the differential explicit forms of the operators are calculated as

$$\begin{aligned}
\mathcal{K}_{\pm\mp}^{N;d';q} = \mathrm{e}^{\mp\mathrm{i}\phi}\Bigg[&\mp\left(q+\left(d'+q+\frac{1}{2}\pm\frac{1}{2}\right)\cos\theta\right)\frac{\partial}{\partial\theta} \mp \mathrm{i}\cos\theta\frac{\partial^2}{\partial\theta\partial\phi} - \left(\frac{1}{\sin\theta}+\sin\theta\right)\frac{\partial^2}{\partial\phi^2} \\
&+\mathrm{i}\left(2\left(d'+q+\frac{1}{2}\pm\frac{1}{2}\right)\sin\theta + \frac{2q+d'+\frac{1}{2}\pm\frac{1}{2}}{\sin\theta}\right)\frac{\partial}{\partial\phi} \\
&- q\frac{q+\left(d'+q+\frac{1}{2}\pm\frac{1}{2}\right)\cos\theta}{\sin\theta}(1-\cos\theta) \\
&+\left(d'+q+\frac{1}{2}\pm\frac{1}{2}\right)\left(d'+2q+\frac{1}{2}\pm\frac{1}{2}\right)\sin\theta\Bigg]
\end{aligned} \tag{42a}$$

$$\begin{aligned}
\mathcal{K}_{\pm\mp}^{S;d';q} = \mathrm{e}^{\mp\mathrm{i}\phi}\Bigg[&\mp\left(q+\left(d'-q+\frac{1}{2}\pm\frac{1}{2}\right)\cos\theta\right)\frac{\partial}{\partial\theta} \mp \mathrm{i}\cos\theta\frac{\partial^2}{\partial\theta\partial\phi} - \left(\frac{1}{\sin\theta}+\sin\theta\right)\frac{\partial^2}{\partial\phi^2} \\
&+\mathrm{i}\left(2\left(d'-q+\frac{1}{2}\pm\frac{1}{2}\right)\sin\theta + \frac{d'-2q+\frac{1}{2}\pm\frac{1}{2}}{\sin\theta}\right)\frac{\partial}{\partial\phi} \\
&- q\frac{q+\left(d'-q+\frac{1}{2}\pm\frac{1}{2}\right)\cos\theta}{\sin\theta}(1+\cos\theta) \\
&+\left(d'-q+\frac{1}{2}\pm\frac{1}{2}\right)\left(d'-2q+\frac{1}{2}\pm\frac{1}{2}\right)\sin\theta\Bigg]
\end{aligned} \tag{42b}$$

$$\mathcal{K}_3^{N(S);q} = -\mathrm{i}\frac{\partial}{\partial\phi} \mp q. \tag{42c}$$

According to equation (15), the bases of given Hilbert subspaces $\mathcal{H}_-^{N(S);d'}$ satisfy the following operator equality:

$$\begin{aligned}
\frac{\partial^2}{\partial\theta^2} = &-\cot\theta\frac{\partial}{\partial\theta} - \cot^2\theta\frac{\partial^2}{\partial\phi^2} - \mathrm{i}\left(\frac{2q(\cos\theta\mp 1)}{\sin^2\theta} + 2d' \pm 2q + 1\right)\frac{\partial}{\partial\phi} + 2q^2\frac{1\mp\cos\theta}{\sin^2\theta} \\
&-(d'\pm q)(d'\pm q+1).
\end{aligned} \tag{43}$$

Thus, applying this equality to the differential form of the commutator $\left[\mathcal{K}_{-+}^{N(S);q;d'}, \mathcal{K}_{+-}^{N(S);q;d'}\right]$, the nonlinear commutation relation associated with these operators is obtained as follows:

$$\begin{aligned}\left[\mathcal{K}_{-+}^{N(S);q;d'}, \mathcal{K}_{+-}^{N(S);q;d'}\right] &= -16\left(\mathcal{K}_3^{N(S);q}\right)^3 + 18(2d'+1)\left(\mathcal{K}_3^{N(S);q}\right)^2 - 2(13d'(d'+1)\\ &\quad - 4q^2 + 5)\mathcal{K}_3^{N(S);q} + 3d'^2\left(2d'+3\right) + d'(7-4q^2) + 2(1-q^2),\\ \left[\mathcal{K}_3^{N(S);q}, \mathcal{K}_{-+}^{N(S);q;d'}\right] &= +\mathcal{K}_{-+}^{N(S);q;d'},\\ \left[\mathcal{K}_3^{N(S);q}, \mathcal{K}_{+-}^{N(S);q;d'}\right] &= -\mathcal{K}_{+-}^{N(S);q;d'}.\end{aligned} \tag{44}$$

The second and third relations of equations (44) are directly obtained by using equations (42). The Casimir operator of the deformed $su(1, 1)$ algebra,

$$\begin{aligned}\mathcal{C}^{N(S);q;d'} &= \mathcal{K}_{-+}^{N(S);q;d'}\mathcal{K}_{+-}^{N(S);q;d'} - 4\left(\mathcal{K}_3^{N(S);q}\right)^4 + 2\left(7+6d'\right)\left(\mathcal{K}_3^{N(S);q}\right)^3 - (13d'^2 + 31d'\\ &\quad - 4q^2 + 18)\left(\mathcal{K}_3^{N(S);q}\right)^2 + (6d'^3 + 22d'^2 + 2d'(13-2q^2) - 6q^2 + 10)\mathcal{K}_3^{N(S);q},\end{aligned} \tag{45}$$

satisfies the following eigenvalue equations on the monopole harmonics belonging to the Hilbert subspace $\mathcal{H}_-^{N(S);d'}$:

$$\mathcal{C}^{N(S);q;d'} Y_{d'-mm}^{N(S);q}(\theta,\phi) = (d'+2)(d'+1)(d'+1+q)(d'+1-q)Y_{d'-mm}^{N(S);q}(\theta,\phi). \tag{46}$$

Thus, relations (41) show that the Hilbert subspaces $\mathcal{H}_-^{N(S);d'}$ are $(l+m)$-integer negative irreducible representations for the deformed algebra of $su(1, 1)$ as relations (44). From the comparison of (37) with (46), we conclude that by changing the parameter d to $-(d'+1)$ the positive representations are transformed to negative ones.

The monopole harmonics $Y_{ss}^{N(S);q}(\theta,\phi)$ and $Y_{s+1s}^{N(S);q}(\theta,\phi)$ as the highest states of two different types of the Hilbert subspaces $\mathcal{H}_-^{N(S);2s}$ and $\mathcal{H}_-^{N(S);2s+1}$ are annihilated by the operators $\mathcal{K}_{-+}^{N(S);q;2s}$ and $\mathcal{K}_{-+}^{N(S);q;2s+1}$, respectively:

$$\mathcal{K}_{-+}^{N(S);q;2s} Y_{ss}^{N(S);q}(\theta\phi) = 0, \tag{47a}$$

$$\mathcal{K}_{-+}^{N(S);q;2s+1} Y_{s+1s}^{N(S);q}(\theta\phi) = 0, \tag{47b}$$

which have the following solutions:

$$Y_{ss}^{N(S);q} = \frac{(-1)^s\,\mathrm{e}^{\mathrm{i}(s\pm q)\phi}}{2^{s+\frac{3}{2}}\sqrt{\pi}}\sqrt{\frac{\Gamma(2s+2)}{\Gamma(s+q+1)\Gamma(s-q+1)}}(1-\cos\theta)^{\frac{s+q}{2}}(1+\cos\theta)^{\frac{s-q}{2}}, \tag{48a}$$

$$\begin{aligned}Y_{s+1s}^{N(S);q} &= \frac{(-1)^s\,\mathrm{e}^{\mathrm{i}(s\pm q)\phi}}{2^{s+1}\sqrt{\pi}}\sqrt{\frac{(2s+3)\Gamma(2s+2)}{\Gamma(s+q+2)\Gamma(s-q+2)}}(1-\cos\theta)^{\frac{s+q}{2}}(1+\cos\theta)^{\frac{s-q}{2}}\\ &\quad\times((s+1)\cos\theta + q).\end{aligned} \tag{48b}$$

The repeated application of relation (41a) for $d' = 2s$ and $d' = 2s+1$ yields

$$\begin{aligned}Y_{2s-mm}^{N(S);q}(\theta,\phi) &= \sqrt{\frac{(4s-2m+1)\Gamma(s+q+1)\Gamma(s-q+1)}{(2s+1)\Gamma(2s-2m+1)\Gamma(2s-m+q+1)\Gamma(2s-m-q+1)}}\\ &\quad\times\left[\mathcal{K}_{+-}^{N(S);q;2s}\right]^{s-m} Y_{ss}^{N(S);q}(\theta,\phi),\end{aligned} \tag{49a}$$

$$Y^{N(S);q}_{2s-m+1m}(\theta,\phi) = \sqrt{\frac{(4s-2m+3)\Gamma(s+q+2)\Gamma(s-q+2)}{(2s+3)\Gamma(2s-2m+2)\Gamma(2s-m+q+2)\Gamma(2s-m-q+2)}}$$
$$\times\left[\mathcal{K}^{N(S);q;2s+1}_{+-}\right]^{s-m} Y^{N(S);q}_{s+1s}(\theta,\phi), \tag{49b}$$

where $m = s, s-1, s-2, \dots$. Therefore, we have shown that the set of all monopole harmonics can be considered by two different methods as the union of irreducible representation subspaces of the deformed $su(1,1)$ algebra. We have also concluded that the deformed $su(1,1)$ algebra as an internal symmetry for the surface bound states of the magnetic monopole problem corresponds exactly to a spectrum-generating algebra based on the simultaneous quantization of both azimuthal and magnetic numbers l and m, respectively.

Finally, it should be emphasized that we can use d and d' as the deformation parameters and obtain the undeformed $su(1,1)$ algebra as a continuous limit of the commutation rules (35) and (44), similar to what has been done in [16]. Let us define new generators for the nonlinear algebras as follows:

$$\mathcal{M}^{N(S);q}_{\pm} := \frac{1}{\mathrm{d}}\mathcal{K}^{N(S);q;d}_{\pm\pm}, \qquad \mathcal{M}^{N(S);q}_{3} := \mathcal{K}^{N(S);q}_{3} + d + \frac{1}{2}, \tag{50a}$$

$$\mathcal{N}^{N(S);q}_{\pm} := \frac{1}{d'}\mathcal{K}^{N(S);q;d'}_{\mp\pm}, \qquad \mathcal{N}^{N(S);q}_{3} := \mathcal{K}^{N(S);q}_{3} - d' - \frac{1}{2}. \tag{50b}$$

If we compute the commutators for the cubic algebras in the new bases, and take the limits d and d' to infinity, then we obtain commutation relations corresponding to the $su(1,1)$ Lie algebra as

$$\left[\mathcal{M}^{N(S);q}_{+}, \mathcal{M}^{N(S);q}_{-}\right] = -2\mathcal{M}^{N(S);q}_{3}, \qquad \left[\mathcal{M}^{N(S);q}_{3}, \mathcal{M}^{N(S);q}_{\pm}\right] = \pm\mathcal{M}^{N(S);q}_{\pm}, \tag{51a}$$

$$\left[\mathcal{N}^{N(S);q}_{+}, \mathcal{N}^{N(S);q}_{-}\right] = -2\mathcal{N}^{N(S);q}_{3}, \qquad \left[\mathcal{N}^{N(S);q}_{3}, \mathcal{N}^{N(S);q}_{\pm}\right] = \pm\mathcal{N}^{N(S);q}_{\pm}. \tag{51b}$$

It is now straightforward to show that the Casimir operators for the $su(1,1)$ Lie algebras (51a) and (51b), i.e.

$$\mathcal{C}^{N(S);q}_{\mathcal{M}} = \mathcal{M}^{N(S);q}_{+}\mathcal{M}^{N(S);q}_{-} - \left(\mathcal{M}^{N(S);q}_{3}\right)^2 + \mathcal{M}^{N(S);q}_{3}, \tag{52a}$$

$$\mathcal{C}^{N(S);q}_{\mathcal{N}} = \mathcal{N}^{N(S);q}_{+}\mathcal{N}^{N(S);q}_{-} - \left(\mathcal{N}^{N(S);q}_{3}\right)^2 + \mathcal{N}^{N(S);q}_{3}, \tag{52b}$$

can be followed from the deformed Casimir operators (36) and (45) by the limiting processes $d\to\infty$ and $d'\to\infty$, respectively. Note that in this case, all constant terms in the deformed Casimir operators must be subtracted before the limits d and d' to infinity are considered.

5. Conclusion

In this paper, positive and negative integer irreducible representations of a deformed $su(1,1)$ algebra on the space of monopole harmonics are obtained for given values of $l-m$ and $l+m$, respectively. They are realized by the ladder differential operators which change l and m by one unit. The positive and negative irreducible nonunitary representations (32) and (41) of the deformed $su(1,1)$ algebra are the output of the ladder symmetries (17a),(17b) and (27a),(27b). For this reason, we call them 'internal symmetries' which are in turn considered as new spectrum-generating symmetries. These realizations of the deformed algebra are therefore understood as an internal symmetry for the Hilbert space corresponding to all monopole harmonics. Note that the quantization of both azimuthal and magnetic numbers l and m of the monopole harmonics are used jointly. Therefore, the nonlinear commutation relations (35) and (44) not only quantize the indices l and m as mentioned above, but also such a realization of the deformed $su(1, 1)$ algebra is responsible for the generation of monopole harmonics.

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